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Combining rotation curves and gravitational lensing: How to measure the equation of state of dark matter in the galactic halo

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ABSTRACT

We argue that combined observations of galaxy rotation curves and gravitational lensing not only allow the deduction of a galaxy's mass profile, but also yield information about the pressure in the galactic fluid. We quantify this statement by enhancing the standard formalism for rotation curve and lensing measurements to a first post-Newtonian approximation. This enhanced formalism is compatible with currently employed and established data analysis techniques, and can in principle be used to reinterpret existing data in a more general context. The resulting density and pressure profiles from this new approach can be used to constrain the equation of state of the galactic fluid, and therefore might shed new light on the persistent question of the nature of dark matter.

Key words: equation of state – gravitational lensing – methods: data analysis – galaxies: halos – galaxies: kinematics and dynamics – dark matter.

1 INTRODUCTION

One of the most compelling issues of modern astrophysics is the open question concerning the nature of the dark matter which dominates the gravitational field of individual galaxies and galaxy clusters. [See for instance Persic et al. (1996), Borriello & Salucci (2001), and Salucci & Borriello (2003).] While the current consensus in the astrophysics community is to advocate the cold dark matter (CDM) paradigm, no *direct* observations of the equation of state have been carried out to confirm this widely adopted assumption. Efforts to confirm this assumption include attempts to detect elementary particles that have been suggested as cold dark matter candidates. However, experiments that aim (for instance) to detect massive axions with Earth-based detectors (Particle Data Group: Eidelman et al. 2004, §22.2.2) do not yet yield a positive result.

A different approach to analysing the nature of dark matter has been suggested by Bharadwaj & Kar (2003) who first proposed that combined measurements of rotation curves and gravitational lensing could be used to determine the equation of state of the galactic fluid. Whereas their analysis made particular assumptions on the form of the rotation curve, and is restricted to a certain type of equation of state, herein we provide a general formalism that allows us to deduce the density and pressure profiles without any prior assumptions about their shape or the equation of state.

Analytic galaxy halo models that predict a significant amount of pressure or tension in the dark matter fluid include "string fluid" (Soleng 1995), or some variations of scalar field dark matter (SFDM). See for instance Schunck (1999), Matos, Guzmán & Ureña-López (2000), Matos, Guzmán & Núñez (2000), Peebles (2000), and Arbey, Lesgourgues & Salati (2003). Our method provides a means of observing, or at least constraining, the pressure distribution in a galactic halo. Therefore it is in principle able to give evidence for or against specific proposed dark matter candidates.

The key point is that in general relativity, density and pressure *both* contribute to generating the gravitational field *separately.* Furthermore, the perception of this gravitational field depends on the velocity of probe particles. These effects become especially important when one compares rotation curve and gravitational lensing measurements, where the probe particles are fundamentally different: interstellar gas or stars at subluminal velocities for rotation curves, and photons which travel at the speed of light for lensing measurements. Our formalism accounts for these crucial differences between the probe particles, and relates observations of both kinds to the the density and pressure profile of the host galaxy. Although we (mainly) consider static spherically symmetric galaxies in a first post-Newtonian approximation, the basic concept is fundamental and can be ex-

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tended to more general systems with less symmetry. A suitable framework for considering most exotic weak gravity scenarios is provided by the effective refractive index tensor, as introduced by Boonserm, Cattoen, Faber, Visser & Weinfurtner (2005).

The present approach might also help to shed some light on prevailing problems that arise when combining rotation curve and lensing observations. For example, an unresolved issue exists when measuring the Hubble constant from the time delay between gravitationally lensed images: Using the standard models for matter distribution in the lens galaxy, the resulting Hubble constant is either too low compared to its value from other observations, or the dark matter halo must be excluded from the galaxy model to obtain the commonly accepted value of H_0 (Kochanek & Schechter 2004). A possible explanation of this trend might lie in a disregarded pressure component of the dark matter halo.

We organise this article in the following manner: First we introduce the minimal necessary framework of general relativity concepts, and point out the important conditions required to obtain the Newtonian gravity limit. Next, we elaborate on the post-Newtonian extension of the currently employed rotation curve and gravitational lensing formalisms. Consequently, we show how to combine rotation curve and lensing measurements to make inferences about the density and pressure profile of the observed galaxy. We then examine how noticeable the effects of non-negligible pressure could be in the measurements. Lastly, we discuss how the formalism adapts to non-spherically symmetric galaxies and comment on the current observational situation and issues arising with the new formalism.

2 GENERAL RELATIVITY FRAMEWORK

In general relativity the motion of a probe particle is given as the geodesic of a curved space-time whose curvature is generated by matter or more generally speaking, stress-energy. The static and approximately spherically symmetric gravitational field of a galaxy is represented by the space-time metric (Misner, Thorne & Wheeler 1973, §23.2)

$$\mathrm{d}s^2 = -e^{2\Phi(r)}\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2m(r)/r} + r^2\,\mathrm{d}\Omega^2\,, \tag{1}$$

which is completely determined by the two metric functions $\Phi(r)$ and $m(r)$. These coordinates (t, r, θ, φ) are called *curvature coordinates* and

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2 \tag{2}$$

represents the geometry of a unit sphere. Invoking the Einstein field equations with the most general static and spherically symmetric stress-energy tensor gives the relation between the metric functions and the density and pressure profiles:

$$8\pi\,\rho(r) = \frac{2\,m'(r)}{r^2}\,; \tag{3}$$

$$8\pi\,p_r(r) = -\frac{2}{r^2}\left[\frac{m(r)}{r} - r\,\Phi'(r)\left(1-\frac{2\,m(r)}{r}\right)\right]; \tag{4}$$

$$8\pi\,p_t(r) = -\frac{1}{r^3}\left[m'(r)\,r - m(r)\right]\left[1 + r\,\Phi'(r)\right] + \left(1-\frac{2\,m(r)}{r}\right)\left[\frac{\Phi'(r)}{r} + \Phi'(r)^2 + \Phi''(r)\right], \tag{5}$$

where $\rho(r)$ is the energy-density profile, and $p_r(r)$ and $p_t(r)$ denote the profiles of the principal pressures in the radial and transverse directions. Note that we use geometrical units ($c = 1$, $G_\mathrm{N} = 1$) unless otherwise mentioned. Hence, if the metric functions $\Phi(r)$ and $m(r)$ are given by observations, one can infer the density and pressure profiles. For a perfect fluid, one would expect $p = p_r = p_t$. From (3) the physical interpretation of $m(r)$ as the total mass-energy within a sphere of radius r becomes clear.

2.1 The Newtonian limit

Standard Newtonian physics is obtained in the limit of general relativity where (Misner et al. 1973, §17.4):

(i) the gravitational field is weak ($2m/r \ll 1$, $2\Phi \ll 1$),
(ii) the probe particle speeds involved are slow compared to the speed of light and
(iii) the pressures and matter fluxes are small compared to the mass-energy density.

While there is no doubt that in a galaxy condition (i) is satisfied everywhere apart from the central region (Schödel et al. 2002), condition (ii) only holds for rotation curves and not for gravitational lensing. Finally, condition (iii) is related to the open question about the fundamental nature of dark matter. Hence, the possibility of dark matter being a high pressure fluid, or some sort of unknown field with high field tensions, cannot be excluded *a priori*.

It is a standard result that condition (i) is enough to deduce that the gravitational potential $\Phi(r)$ is generated by the tt-component of the Ricci tensor (Misner et al. 1973):

$$\boldsymbol{\nabla}^2\Phi \approx \mathbf{R}_{tt}\,, \tag{6}$$

which on invoking the Einstein equations for $\mathbf{R}_{tt}$ becomes

$$\boldsymbol{\nabla}^2\Phi \approx 4\pi\,(\rho + p_r + 2p_t)\,. \tag{7}$$

Therefore, the metric function $\Phi(r)$ can be interpreted as the Newtonian gravitational potential Φ_N if and only if the pressures of the galactic fluid are negligible, i.e. if condition (iii) holds:

$$\boldsymbol{\nabla}^2\Phi_\mathrm{N} = 4\pi\,\rho\,. \tag{8}$$

It is now quite obvious from (7) that the gravitational field is highly sensitive to the pressure if density and pressure are of the same order of magnitude.

3 ROTATION CURVES

For the regime of rotation curve measurements, both conditions (i) and (ii) apply. In this case, the geodesic equations of the metric (1) reduce to (Misner et al. 1973)

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \approx -\boldsymbol{\nabla}\Phi\,, \tag{9}$$

where $\boldsymbol{r}$ denotes the position vector of a probe particle. Equation (9) is equivalent to the Newtonian formulation of gravity, except for the general relativistic potential Φ which replaces the Newtonian potential Φ_N.

Measurements of rotation curves are carried out by observing the Doppler shift in the emission lines of the light emitting probe particles. In a general relativistic context, the observed shift in wavelength is not exclusively due to Doppler effects of the moving probe particles, but also depends on the gravitational redshift which arises as the photons climb out of the gravitational potential well.

It has been shown for edge-on galaxies that the total wavelength shift[1] $z_{\pm}(r)$ of an emission line of a probe particle at radius r is given by (Nucamendi, Salgado & Sudarsky 2001; Lake 2004; Faber 2006)

$$1 + z_{\pm}(r) = \frac{1}{\sqrt{1 - r\,\Phi'(r)}} \left(\frac{1}{e^{\Phi(r)}} - \frac{\pm|b|\,\sqrt{r\,\Phi'(r)}}{r} \right) , \quad (10)$$

where prime denotes the derivative with respect to r, $' = \mathrm{d}/\mathrm{d}r$ and b is the impact parameter. z_+ is the wavelength shift of an approaching particle and z_- that of a receding particle.

The impact parameter is equivalent to the apparent distance between the galactic centre and the emitting particle, once one takes notice of light bending effects. However, in the weak gravity regime of galaxies, where flat space is a suitable approximation, one finds

$$|b| = r + \mathcal{O}[\Phi] \quad (11)$$

for particles whose position vector $\boldsymbol{r}$ (with respect to the galactic centre) is perpendicular to the observer's line of sight (Lake 2004). Thus, with the additional weak field assumption $r\,\Phi'(r) \ll 1$, equation (10) can be written as

$$1 + z_{\pm}(r) = 1 \mp \sqrt{r\,\Phi'(r)} + \mathcal{O}\big[\Phi, r\,\Phi'\big] , \quad (12)$$

or equivalently,

$$z_{\pm}^2 = r\,\Phi'(r) + \mathcal{O}\big[\Phi^2, (r\,\Phi')^{3/2}, \Phi\sqrt{r\,\Phi'}\big] . \quad (13)$$

Comparing this expression to the Doppler shift in Newtonian gravity,

$$\frac{v^2}{c^2} = z_N^2 = r\,\Phi'_N(r) , \quad (14)$$

we conclude that for small particle speeds $v \ll c$, i.e. condition (ii), the observation of $z_{\pm}$ in edge-on galaxies is in first order equivalent to the Doppler redshift in Newtonian gravity, when Φ_{N} is substituted by Φ. This also justifies the previous assumption $r\,\Phi'(r) \ll 1$.

For galaxies of arbitrary orientation it is more tedious to obtain this result, but in a similar fashion, it can also be shown that the Doppler shift in wavelength is the dominant contribution to the observed total redshift (Faber 2006).

Therefore, the usual techniques for obtaining the potential Φ_{RC} from rotation curve measurements can be employed if one keeps in mind that the motion of the observed particles is not governed by the Newtonian gravitational potential Φ_{N}, but by its general relativistic generalisation Φ:

$$\Phi_{\mathrm{RC}} = \Phi \neq \Phi_{\mathrm{N}} . \quad (15)$$

If one assumes condition (iii), the density ρ is related to Φ_{RC} by (8). In the general case, however, the interpretation of the mass which is inferred by rotation curve measurements, $m_{\mathrm{RC}}(r)$, can be obtained from (7):

$$m_{\mathrm{RC}}(r) = r^2\,\Phi'_{\mathrm{RC}} \approx 4\pi \int (\rho + p_r + 2p_t)\, r^2\,\mathrm{d}r . \quad (16)$$

Therefore, in the general case, we call $m_{\mathrm{RC}}(r) \neq m(r)$ the *pseudo-mass* determined by rotation curve measurements.

[1] The wavelength shift that arises from the systemic velocity of the galaxy is not considered here, and Faber (2006) has shown that this does not change the result presented in this context.

4 GRAVITATIONAL LENSING

A fundamentally different approach of measuring the gravitational field of a galaxy is gravitational lensing. Here, the observable photons are not only conveying the information about the gravitational field to us, they also act as probe particles themselves. Hence, condition (ii) is naturally not satisfied for gravitational lensing observations. Consequently, the equations of motion for photons *do not* simplify to (9), as is the case for rotation curves. Instead, the geodesic equations for photons have to be solved exactly to understand the influence of the gravitational field, as it is described by both metric functions, $\Phi(r)$ *and* $m(r)$. Fortunately, for certain spacetimes, such as e.g. (1), it is possible to characterise the entire trajectory of light rays with a single effective refractive index $n(r)$.

4.1 Fermat's principle and the effective refractive index

Fermat's principle of shortest optical paths also applies to the geodesic trajectories of 4-dimensional curved spacetime (Kline & Kay 1965; Misner et al. 1973). This description of light rays in a gravitational field is equivalent to classical optics in a transparent medium with a continuous refractive index n, where Fermat's principle is formulated as the vanishing of the first variation of the optical length between two points, q_1 and q_2, on the trajectory:

$$\delta \int_{q_1}^{q_2} n(\tilde{r})\, \left[\mathrm{d}\tilde{r}^2 + \tilde{r}^2\,\mathrm{d}\Omega^2\right] = 0 . \quad (17)$$

By transforming the curvature coordinates of the spacetime (1) to so called *isotropic coordinates* (Perlick 2004),

$$\mathrm{d}s^2 = e^{2\Phi(\tilde{r})}\,\left\{-\mathrm{d}t^2 + n(\tilde{r})^2\left[\mathrm{d}\tilde{r}^2 + \tilde{r}^2\,\mathrm{d}\Omega^2\right]\right\} , \quad (18)$$

we introduce the scalar effective refractive index $n(\tilde{r})$ of a static spherically symmetric gravitational field. By direct comparison of (1) and (18), we find a differential equation that relates the $\tilde{r}$-coordinate of the isotropic coordinates to the r-coordinate of the curvature coordinates,

$$\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r} = \frac{\tilde{r}}{r\sqrt{1 - 2m(r)/r}} , \quad (19)$$

and the refractive index,

$$n(\tilde{r}) = \frac{r}{\tilde{r}}\, e^{-\Phi(r)} . \quad (20)$$

Since condition (i) is satisfied for the region we are interested in, we can Taylor expand and formally integrate (19) under appropriate boundary conditions and find

$$\tilde{r} = r\,\exp\left\{\int \frac{m(r)}{r^2}\,\mathrm{d}r + \mathcal{O}\left[\left(\frac{2m}{r}\right)^2\right]\right\} , \quad (21)$$

which, inserted into (20), gives

$$n(\tilde r) = \exp\left\{ -\Phi[r(\tilde r)] - \int \frac{m[r(\tilde r)]}{r(\tilde r)^2}\frac{\mathrm{d}r}{\mathrm{d}\tilde r}\,\mathrm{d}\tilde r + \mathcal{O}\left[\left(\frac{2m}{r(\tilde r)}\right)^2\right]\right\}, \tag{22}$$

where $r(\tilde r)$ is given by the inverse of (21). Since $\tilde r = r + \mathcal{O}[2m/r]$, the radii in both sets of coordinates are interchangeable to the desired order and hence, we can also give the refractive index as a function of the curvature coordinate r directly:

$$n(r) = 1 - \Phi(r) - \int \frac{m(r)}{r^2}\,\mathrm{d}r + \mathcal{O}\left[\left(\frac{2m}{r}\right)^2, \frac{2m}{r}\,\Phi, \Phi^2\right]. \tag{23}$$

This effective refractive index entirely determines the trajectory of a light ray, i.e. the probe particles of gravitational lensing. Hence, it is the *only* possible observable of gravitational lensing. We note that the refractive index contains two distinct ingredients, the potential part, $\Phi(r)$, and the integral over the mass-function, $\int 2m(r)/r^2\,\mathrm{d}r$.

At this point, we conclude that since gravitational lensing observations yield $n(r)$ and rotation curve measurements yield $\Phi(r)$, combined observations of $n(r)$ and $\Phi(r)$ allow the separate deduction of $\Phi(r)$ and $m(r)$, and therefore describe the gravitational field of a galaxy in a general relativistic sense, without any prior assumptions. The fundamental principle is that the perception of the gravitational field by probe particles depends on the speed of the probe particles, which manifests itself in the difference of observables $n(r) \neq \Phi(r)$.

For convenience and comparability, we define the lensing potential as

$$2\,\Phi_{\mathrm{lens}}(r) = \Phi(r) + \int \frac{m(r)}{r^2}\,\mathrm{d}r\,, \tag{24}$$

so that

$$n(r) = 1 - 2\,\Phi_{\mathrm{lens}}(r) + \mathcal{O}\left[\Phi_{\mathrm{lens}}^2\right]. \tag{25}$$

4.2 Gravitational lensing formalism

The standard formalism of gravitational lensing in weak gravitational fields is based on the superposition of the deflection angles of many infinitesimal point masses (Schneider, Ehlers & Falco 1992, §4.3).

In general relativity, a point mass M is described by the Schwarzschild exterior metric,

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2M/r} + r^2\,\mathrm{d}\Omega^2\,, \tag{26}$$

which inserted into (23) gives the effective refractive index

$$n(r) = 1 + \frac{M}{r} - \int \frac{M}{r^2}\,\mathrm{d}r + \mathcal{O}\left[\left(\frac{2M}{r}\right)^2\right] \tag{27}$$

$$= 1 + \frac{2M}{r} + \mathcal{O}\left[\left(\frac{2M}{r}\right)^2\right]. \tag{28}$$

In the Newtonian limit, this is generally identified with the Newtonian potential,

$$n(r) = 1 - 2\,\Phi_{\mathrm{N}}(r)\,, \tag{29}$$

whereas from (27) it is clear that in the general case, the refractive index is only partially specified by the potential term. That the mass term of the refractive index of a point mass is identical to the potential term is a special case of the Schwarzschild metric. Keeping this in mind, we proceed to outline the current formalism.

For a point mass, the angular displacement of the gravitationally lensed image in the lens plane can be calculated from the refractive index (28) (Schneider et al. 1992):

$$\hat{\boldsymbol{\alpha}} = 4M\,\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\,, \tag{30}$$

where $\boldsymbol{\xi}$ is the vector that connects the lensed image and the centre of the lens in the 2-dimensional lens plane. Since the extent of a lensing galaxy's mass distribution is small compared to the distance between the light emitting background object and the lens, as well as compared to the distance between the lensing galaxy and the observer, one assumes the deflecting mass distribution to be geometrically thin. Therefore, the volume density ρ of the lensing galaxy can be projected onto the so-called lens plane, resulting in the surface density $\Sigma(\boldsymbol{\xi})$ which describes the mass distribution within the lens plane (Schneider 1985).

The total deflection angle of a lensing mass with finite extent is then said to be given by the superposition of all small angles (30) due to the infinitesimal masses in the lens plane (Schneider et al. 1992):

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = 4\int \Sigma(\boldsymbol{\xi}')\,\frac{\boldsymbol{\xi}-\boldsymbol{\xi}'}{|\boldsymbol{\xi}-\boldsymbol{\xi}'|^2}\,\mathrm{d}^2\xi'\,. \tag{31}$$

This equation is the foundation of the gravitational lensing formalism as introduced by Schneider (1985), and Blandford & Narayan (1986).

Since this formalism is based on the assumption that the total angle of deflection is caused by the superposition of point masses – without the notion of pressure at all – it automatically assumes that the underlying lensing potential is Newtonian, i.e. $\Phi_{\mathrm{lens}} = \Phi_{\mathrm{N}}$. Hence, the lensing potential and the naively inferred density- or mass-distribution are related by (8):

$$\boldsymbol{\nabla}^2\Phi_{\mathrm{lens}}(r) = 4\pi\,\rho_{\mathrm{lens}}(r) \tag{32}$$

which implies

$$\Phi_{\mathrm{lens}}(r) = \int \frac{m_{\mathrm{lens}}(r)}{r^2}\,\mathrm{d}r\,. \tag{33}$$

However, as we argued previously, the lensing potential Φ_{lens} is the fundamental observable, and not the density ρ which was used to construct the formalism. Therefore, for the general case that does not assume (iii), i.e. $\Phi_{\mathrm{lens}} \neq \Phi_{\mathrm{N}}$, we note that

$$\rho_{\mathrm{lens}}(r) \neq \rho(r) \quad \text{and} \quad m_{\mathrm{lens}}(r) \neq m(r)\,. \tag{34}$$

Instead, the deduced mass distribution $m_{\mathrm{lens}}(r)$ has to be considered as a *pseudo-mass* similar to that of rotation curve measurements. Its physical interpretation can be deduced from the definition of the lensing potential (24):

$$m_{\mathrm{lens}}(r) = \frac{1}{2}\,m_{\mathrm{RC}}(r) + \frac{1}{2}\,m(r) \tag{35}$$

$$\approx 4\pi\int\left[\rho + \frac{1}{2}\,(p_r + 2p_t)\right] r^2\,\mathrm{d}r\,. \tag{36}$$

5 BRINGING ROTATION CURVES AND GRAVITATIONAL LENSING TOGETHER

We showed in the previous sections that the potentials obtained from rotation curve and lensing observations, $\Phi_{\rm RC}$ and $\Phi_{\rm lens}$, do not agree in the general case,

$$\Phi_{\rm RC}(r) = \Phi(r)\,, \tag{37}$$

$$\Phi_{\rm lens}(r) = \frac{1}{2}\,\Phi(r) + \frac{1}{2}\int \frac{m(r)}{r^2}\,{\rm d}r\,, \tag{38}$$

but only in the Newtonian limit, where condition (iii) holds, in which case $\Phi_{\rm RC} = \Phi_{\rm lens} = \Phi_{\rm N}$. Since this is the standard assumption for interpreting rotation curve and lensing data, the results of these observations are often reported as mass distributions instead of potentials. Under the Newtonian assumption, the mass and the potential are related by a field equation of the form (8). In the general case, this leads to the definition of the distinct pseudo-masses,

$$m_{\rm RC}(r) = r^2\,\Phi'(r)\,, \tag{39}$$

$$m_{\rm lens}(r) = \frac{1}{2}\,r^2\,\Phi'(r) + \frac{1}{2}\,m(r)\,, \tag{40}$$

which describe the observations equivalently to the potentials (37) and (38). Equations (39) and (40) can easily be inverted to give the metric functions $\Phi'(r)$ and $m(r)$,

$$\Phi'(r) = \frac{m_{\rm RC}(r)}{r^2}\,, \tag{41}$$

$$m(r) = 2\,m_{\rm lens}(r) - m_{\rm RC}(r)\,, \tag{42}$$

which inserted into the field equations of general relativity (3)–(5) yield the density and pressure profiles:

$$4\pi\,r^2\rho(r) = 2\,m'_{\rm lens}(r) - m'_{\rm RC}(r)\,, \tag{43}$$

$$4\pi\,r^2 p_r(r) = 2\,\frac{m_{\rm RC}(r) - m_{\rm lens}(r)}{r} + \mathcal{O}\left[\left(\frac{2\,m}{r}\right)^2\right]\,, \tag{44}$$

$$\begin{aligned} 4\pi\,r^2 p_t(r) &= r\left[\frac{m_{\rm RC}(r) - m_{\rm lens}(r)}{r}\right]' + \mathcal{O}\left[\left(\frac{2\,m}{r}\right)^2\right] \\ &= \frac{r}{2}\left[4\pi\,r^2 p_r(r)\right]' + \mathcal{O}\left[\left(\frac{2\,m}{r}\right)^2\right]\,. \end{aligned} \tag{45}$$

As consistency checks, we note that:

- The Einstein equations in curvature coordinates, (43)–(45), agree to the given order of $2m/r$ with the Einstein equations of the metric in isotropic coordinates, and therefore the approximation $\tilde{r} \approx r$ is valid;
- From the equations (43)–(45) follows that

$$4\pi\,r^2\left[\rho(r) + p_r(r) + 2p_t(r)\right] \approx m'_{\rm RC}(r)\,, \tag{46}$$

and

$$4\pi\,r^2\left[\rho(r) + \frac{1}{2}\left(p_r(r) + 2p_t(r)\right)\right] \approx m'_{\rm lens}(r)\,, \tag{47}$$

and thus these results are consistent with the weak field approximation of the field equations (7), and the interpretations of the pseudo-masses (16) and (36);
- For $m'_{\rm RC}(r) = m'_{\rm lens}(r) = m'(r)$ we find the desired result of the Newtonian limit:

$$4\pi\,r^2\rho(r) = m'(r)\,, \tag{48}$$

$$4\pi\,r^2 p_r(r) = \mathcal{O}\left[\left(\frac{2\,m}{r}\right)^2\right]\,, \tag{49}$$

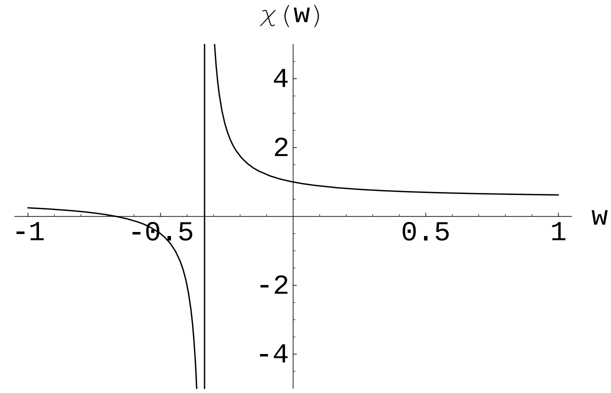

Figure 1. The χ-factor (53) as a function of w in the commonly discussed range $w \in [-1, 1]$. Naturally $\chi(0) = 1$, which corresponds to the Newtonian case, see (48) – (50). At the ends of the plotted range we find $\chi(-1) = 1/4$ and $\chi(1) = 5/8$. The zero-crossing and the first-order pole correspond to $w = -2/3$ and $w = -1/3$ respectively, as is obvious from (53).

$$4\pi\,r^2 p_t(r) = \mathcal{O}\left[\left(\frac{2\,m}{r}\right)^2\right]\,. \tag{50}$$

We conclude that the currently existing formalisms for analysing data from rotation curve and gravitational lensing observations can be used to separately obtain the pseudo-masses $m_{\rm RC}(r)$ and $m_{\rm lens}(r)$, which by (43)–(45) yield the density and pressure profiles in a first post-Newtonian approximation. Furthermore, from the combination

$$4\pi\,r^2\left(p_r + 2p_t\right) \approx 2\left(m'_{\rm RC} - m'_{\rm lens}\right)\,, \tag{51}$$

one can immediately infer that the observed system is Newtonian in the sense of condition (iii) if and only if $m'_{\rm RC}(r) \approx m'_{\rm lens}(r)$. Furthermore, defining the dimensionless quantity

$$w(r) = \frac{p_r(r) + 2p_t(r)}{3\rho(r)} \approx \frac{2}{3}\,\frac{m'_{\rm RC}(r) - m'_{\rm lens}(r)}{2\,m'_{\rm lens}(r) - m'_{\rm RC}(r)} \tag{52}$$

gives a convenient parameter that determines a "measure" of the equation of state.

6 PARAMETERIZING THE SIZE OF THE EFFECT

To get an idea how noticeable the existence of a pressure contribution is likely to be in the measured data, we introduce the χ-factor which we define as the ratio of the derivatives of $m_{\rm lens}$ and $m_{\rm RC}$:

$$\chi[w(r)] = \frac{m'_{\rm lens}(r)}{m'_{\rm RC}(r)} = \frac{2 + 3w(r)}{2 + 6w(r)}\,. \tag{53}$$

This can easily be obtained from rearranging (52).

Using fig. 1, we can now see how the ratio of the slopes of $m_{\rm lens}$ and $m_{\rm RC}$ relates to w. One should especially note that χ is not very different from unity in the vicinity of $w = 0$, making it difficult to detect small pressures. However, for $w \in [1/2, 1]$, which is a range that might plausibly be identified with real-life data (or at the very least with some real-world theoretical prejudices), χ takes values between 62.5% and 70%. For most negative w, specifically for w in the interval $[-1, 0]$, χ is rather distinct from unity, and therefore

should be easily detectable if the quality of the observational data is high.

Also note that in the special case of a position-independent w — so that the "equation of state of the galactic fluid" is constant throughout the observed region — the ratio between $m_{\rm lens}$ and $m_{\rm RC}$ will be the same as between their derivatives:

$$m_{\rm lens}(r) = \chi(w)\, m_{\rm RC}(r)\,. \tag{54}$$

This relation is likely to be more useful since it does not depend on numerically obtained derivatives when one wishes to compare mass profiles. However, it comes at the price of the additional assumption that $w = $ constant.

7 NON-SPHERICAL GALAXIES

Although we chose to present this new formalism using the example of a simple spherically symmetric galaxy, it is easy to show that the fundamental concept behind the formalism is also valid for configurations with less symmetry.

7.1 Rotation curves

The formalism of Newtonian mechanics is usually adopted when data from dynamical observations is examined to determine the shape of the matter distribution of a galaxy. That is, the fundamental equation employed is (9):

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \approx -\boldsymbol{\nabla}\Phi(x,y,z)\,. \tag{55}$$

Note that no particular symmetry is now assumed for the gravitational potential $\Phi(x,y,z)$. The only necessary assumptions are the weakness of the gravitational field [condition (i)] and the slowness of the probe particles [condition (ii)].

In view of the formalism presented herein, one now only needs to realise that the gravitational potential inferred from dynamical observations is generally not the Newtonian potential

$$\Phi_{\rm RC}(x,y,z) = \Phi(x,y,z) \neq \Phi_{\rm N}(x,y,z)\,, \tag{56}$$

and that the valid field equation is [*cf* (7)]:

$$\boldsymbol{\nabla}^2\Phi(x,y,z) \approx 4\pi\left[\rho(x,y,z) + \sum\nolimits_{i=1}^{3} p_i(x,y,z)\right]\,. \tag{57}$$

7.2 Gravitational lensing

The notion of a simple effective refractive index can easily be extended to the class of *conformally static* spacetimes which are also *conformally Euclidean* (Perlick 2004). The metric of this class has no particular spatial symmetry and takes the form:

$$\mathrm{d}s^2 = e^{2\Phi(x,y,z)}\left\{-\mathrm{d}t^2 + n(x,y,z)^2\left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right]\right\}\,. \tag{58}$$

To account for the weakness of the gravitational field [condition (i)], we assume

$$\Phi(x,y,z) \ll 1 \tag{59}$$

and

$$n(x,y,z) = 1 + h(x,y,z) \quad \text{with} \quad h(x,y,z) \ll 1\,. \tag{60}$$

The first Einstein field equation[2] is then

$$\begin{aligned} 4\pi\rho(x,y,z) &= -\boldsymbol{\nabla}^2\Phi(x,y,z) - \boldsymbol{\nabla}^2 h(x,y,z) \\ &\quad + \mathcal{O}\left[\Phi^2, h\Phi, h^2\right]\,. \end{aligned} \tag{61}$$

Inverting this equation yields the effective refractive index

$$\begin{aligned} n(x,y,z) &= 1 - \Phi(x,y,z) - 4\pi\,(\boldsymbol{\nabla}^2)^{-1}\rho(x,y,z) \\ &\quad + \mathcal{O}\left[\Phi^2, h\Phi, h^2\right]\,, \end{aligned} \tag{62}$$

where the constants of integration have been chosen to agree with the special case of spherical symmetry (23), and $(\boldsymbol{\nabla}^2)^{-1}$ is the inverse Laplacian operator. The general non-spherical lensing potential analogous to (24) can be defined as

$$2\,\Phi_{\rm lens}(x,y,z) = \Phi(x,y,z) + 4\pi\,(\boldsymbol{\nabla}^2)^{-1}\rho(x,y,z)\,, \tag{63}$$

so that the non-spherical refractive index is of the same form as (25):

$$n(x,y,z) = 1 - 2\,\Phi_{\rm lens}(x,y,z) + \mathcal{O}\left[\Phi_{\rm lens}^2\right]\,. \tag{64}$$

The corresponding field equation for the lensing potential is

$$\boldsymbol{\nabla}^2\Phi_{\rm lens}(x,y,z) = \frac{1}{2}\boldsymbol{\nabla}^2\Phi(x,y,z) + 2\pi\,\rho(x,y,z) \tag{65}$$

$$= 4\pi\left[\rho(x,y,z) + \frac{1}{2}\sum\nolimits_{i=1}^{3} p_i(x,y,z)\right]\,. \tag{66}$$

7.3 Non-spherical formalism

Combining the non-spherical field equations (57) and (66) yields the density and pressure distributions in absence of any particular spatial symmetry:

$$4\pi\,\rho \;\approx\; 2\,\boldsymbol{\nabla}^2\Phi_{\rm lens}(x,y,z) - \boldsymbol{\nabla}^2\Phi_{\rm RC}(x,y,z) \tag{67}$$

$$4\pi\sum\nolimits_{i=1}^{3} p_i \;\approx\; 2\left[\boldsymbol{\nabla}^2\Phi_{\rm RC}(x,y,z) - \boldsymbol{\nabla}^2\Phi_{\rm lens}(x,y,z)\right] \tag{68}$$

Thus the two observable potentials $\Phi_{\rm RC}(x,y,z)$ and $\Phi_{\rm lens}(x,y,z)$ determine the density and pressure distributions of a non-spherical galaxy to the lowest order in the weak gravitational field represented by the functions $\Phi(x,y,z)$ and $h(x,y,z)$. This is the straightforward extension of the spherically symmetric formalism presented in this paper.

8 OBSERVATIONAL SITUATION

The post-Newtonian formalism we have outlined requires the simultaneous measurement of (pseudo-)density profiles from rotation curve and gravitational lensing observations.

While in principle these profiles do not have to be of the same galaxy, they must be comparable in the sense that they accurately describe "similar" galaxies. For example, weak lensing measurements can be used to statistically infer the (pseudo)-density profile of an "average" galaxy (Brainerd 2004). At the same time, analysing the dynamics of satellite galaxies gives the rotation curve and thus, the corresponding pseudo-density profile, of another "average" galaxy (Brainerd 2004). Whether these two "average" galaxies are comparable or not depends on many factors, such as e.g.

[2] The first Einstein field equation is that which is associated with the tt-component of the Einstein tensor.

the distribution of galaxy morphologies in both samples, the statistical noise, the employed models for the (pseudo)-density distribution, etc. These statistical issues render the fast-growing collection of weak lensing data problematic for our purposes.

On the other hand, combined simultaneous measurements of rotation curves and lensing of individual galaxies are extremely well suited for our formalism. However, while there is a large number of individual rotation curves available ($> 100,000$; Sofue & Rubin 2001), the number of individual "strong" lensing systems with multiple images is rather limited[3] (~ 70; Kochanek et al. 2005). Combined observations are further aggravated by the differing distance scales: Most high quality rotation curves are naturally available for galaxies with a low to intermediate redshift of up to $z \sim 0.4$ (Sofue & Rubin 2001), while gravitational lenses are easier to detect at intermediate to high redshifts ($z \gtrsim 0.4$; Kochanek et al. 2005), since the image separation scales increasingly with the redshift of the lensing galaxy (Schneider 1985; Kochanek & Schechter 2004). Therefore, even for nearby galaxies with existing combined measurements of kinematics and lensing (e.g. 2237+0305 at $z \approx 0.039$ and ESO 325-G004 at $z \approx 0.035$), the lensing data is restricted to the core region, while the rotation curve is only described by few data points in the outer region of the lens galaxy (Barnes et al. 1999; Trott & Webster 2002; Smith et al. 2005). Consequently, the inferred pseudo-mass profiles are available for the same galaxy, but unfortunately at different radii and therefore not comparable.

Although the observational situation makes it currently difficult to employ the formalism presented, the situation is likely to improve in the future when observations with a higher resolution will be carried out – preferably with an emphasis on obtaining high-resolution rotation curves for lensing galaxies that exhibit lensed images at different radii.

9 CONCLUSIONS

We have argued that the standard formalism of rotation curve measurements and gravitational lensing make an *a priori* Newtonian assumption that is based on the CDM paradigm. We introduce a post-Newtonian formalism that does not rely on such an assumption, and furthermore allows one to deduce the density- and pressure-profiles in a general relativistic framework. In this framework, rotation curve measurements provide a pseudo-mass profile $m_{\rm RC}(r)$ and gravitational lensing observations yield a different pseudo-mass profile $m_{\rm lens}(r)$. Combining both pseudo-masses allows one to draw conclusions about the density- and pressure profiles[4] in the lensing galaxy,

$$\rho(r) = \frac{1}{4\pi\, r^2} \left[2\, m'_{\rm lens}(r) - m'_{\rm RC}(r)\right] , \qquad (69)$$

$$p_r(r) + 2p_t(r) \approx \frac{2\, c^2}{4\pi\, r^2} \left[m'_{\rm RC}(r) - m'_{\rm lens}(r)\right] . \qquad (70)$$

In the case of absent or negligible pressure, this could be used to observationally confirm the CDM paradigm of a pressureless galactic fluid. Conversely, if significant pressure is detected, a decomposition of the galaxy morphology would allow new insight into the equation of state of dark matter.

For instance, detailed observation of the recently discovered closest known strong lensing galaxy ESO 325-G004 (Smith et al. 2005) could provide satisfactory data to allow the decomposition of density and pressure of the galactic fluid, as outlined in this article. The system consists of an isolated lensing galaxy at redshift $z \approx 0.035$ with an effective radius of $R_{\rm eff} = 12\farcs5$ and arc-shaped images of the background object at $R \approx 3''$, and possible arc candidates at $R \approx 9''$. Smith et al. (2005) intend to collect more detailed data that hopefully will include extended stellar dynamics and hence, allow for a direct comparison of the rotation curve and lensing data, if the arc candidates at $R \approx 9''$ turn out to contribute to the measurements.

Since the formalism presented is based on a first-order weak field approximation, we suggest that to confirm the findings, one should re-insert the obtained density and pressure profiles into the metric (1). The actual observed quantities can then be extracted numerically for comparison from the exact field equations (3)–(5) and the geodesic equations.

Finally, even though data might not yet be available to constrain the dark matter equation of state noticeably, one should note that the possibility of non-negligible pressure in the galactic fluid introduces a new free parameter into the analysis of combined rotation curve and lensing observations.

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[3] For an up to date list see `http://www.cfa.harvard.edu/glensdata/`.

[4] These formulae are given in SI units, hence the factor of c^2.

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